Baxter Equation for the QCD Odderon

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Abstract

The Hamiltonian derived by Bartels, Kwiecinski and Praszalowicz for the study of high-energy QCD in the generalized logarithmic approximation was found to correspond to the Hamiltonian of an integrable XXX spin chain. We study the odderon Hamiltonian corresponding to three sites by means of the Bethe Ansatz approach. We rewrite the Baxter equation, and consequently the Bethe Ansatz equations, as a linear triangular system. We derive a new expression for the eigenvectors and the eigenvalues, and discuss the quantization of the conserved quantities.

June 1995
T95/081
hep-th/9507056

1 Introduction

The behaviour of hadronic scattering amplitudes at high-energies for fixed transferred momentum t is, with confinement, one of the most interesting problems to be solved in the strong interaction field.

The discovery of QCD gave us the theory for studying this problem. In the framework of perturbative QCD, the resummation of leading logarithmic amplitudes was performed [1] and gave the so-called "perturbative Pomeron", which violates the Froissart bound [2] derived from the analyticity and unitarity of the S-matrix. This is why one needs to take into account subleading terms, so as to restore unitarity.

It is possible to write down an action which takes into account all graphs in the large s limit [3]. But it is not easy to perform calculations in this framework. It turns out that one can consider a subclass of such diagrams where one takes into account only the exchange of a fixed number of reggeons in the t-channel, keeping only the dominant contribution at each level [4]. This model can be formulated as a Schrödinger equation with a two-body interaction Hamiltonian. This Hamiltonian was later shown [5, 6] to belong to the integrable hierarchy of Hamiltonians of the Heisenberg XXX spin-chain with $SL(2,\mathbf{C})$-spin zero. The Bethe Ansatz method was then applied in this framework to diagonalize the Hamiltonian. In this paper we analyze the Baxter equation and eigenvectors which arise in this context. In section (3) we derive general results for the solution $Q_n(\lambda)$ of the Baxter equation for n sites, and more particularly the $n = 2$ and $n = 3$ chains. In section (4) we rewrite the Baxter equation in a new form, more appropriate to the study of the polynomial solutions. And in section (5) we give new expressions for the eigenvectors.

2 The Reggeon Hamiltonian

After Fourier transform, in the two-dimensional impact parameter space, the interaction Hamiltonian $\mathcal{H}_{jk}$ between two reggeons becomes the sum of a holomorphic piece and its anti-holomorphic counterpart. The holomorphic piece has the two equivalent representations [7]:

$$\begin{aligned} H_{jk} &= -P_j^{-1}\log\left(z_{jk}\right)P_j - P_k^{-1}\log\left(z_{jk}\right)P_k - \log\left(P_jP_k\right) - 2\gamma &(1)\\ &= -2\log(z_{jk}) - z_{jk}\log\left(P_jP_k\right)z_{jk}^{-1} - 2\gamma &(2)\end{aligned}$$

where

$$z_{jk} = z_j - z_k\ ,\quad P_j = i\frac{\partial}{\partial z_j}\ ,$$

and γ is the Euler constant. There is also another equivalent form

$$H_{jk} = \sum_{\ell=0}^{\infty}\left(\frac{2\ell+1}{\ell\left(\ell+1\right) - L_{jk}^2} - \frac{2}{\ell+1}\right)\ ,\quad L_{jk}^2 = -z_{jk}^2\frac{\partial^2}{\partial z_j \partial z_k}\ . \tag{3}$$

The Hamiltonian (3) and its antiholomorphic counterpart are clearly invariant under the conformal transformations [8]

$$z_j \longrightarrow z_j^{'} = \frac{az_j + b}{cz_j + d} \;\; , \;\; \overline{z}_j \longrightarrow \overline{z^{'}}_j = \frac{\overline{a}\overline{z}_j + \overline{b}}{\overline{c}\overline{z}_j + \overline{d}} \tag{4}$$

with $ad - bc = \overline{a}\overline{d} - \overline{b}\overline{c} = 1$

The complete reggeon Hamiltonian is given by

$$\mathcal{H}_n = -\frac{\alpha_s}{2\pi} \sum_{n \geq j > k \geq 1} \left(H_{jk} + \overline{H}_{jk} \right) t_j^a t_k^a$$

where the t_j^a are the color matrices for the j^{th} reggeon and α_s is the strong coupling constant.

The eigenstates $\chi_{n,\{q\}}$ are parametrized by a set of quantum numbers $\{q\}$ and the additional coordinates $(z_0, \overline{z}_0)$ which correspond to the center of mass of the compound Reggeon state. In the large-N_c limit, one has

$$t_1^a t_2^a \longrightarrow -N_c \quad \text{for} \quad n = 2 \qquad t_j^a t_k^a \longrightarrow -\frac{N_c}{2} \delta_{k,j+1} \quad \text{for} \quad n \geq 3\,, \tag{5}$$

and $\mathcal{H}_n$ becomes holomorphically separable:[1]

$$\mathcal{H}_n = \frac{\alpha_s N_c}{4\pi} \left(H_n + \overline{H}_n \right) = \frac{\alpha_s N_c}{4\pi} \left(\sum_{j=1}^{n} H_{j,j+1} + \sum_{j=1}^{n} \overline{H}_{j,j+1} \right) \tag{6}$$

One can then look for eigenvectors in the form

$$\chi_{n,\{q\}} \left(\{z_j, \overline{z}_j\}; z_0, \overline{z}_0 \right) = \varphi_{n,\{q\}} \left(\{z_j\}\,; \overline{z}_0 \right) \overline{\varphi}_{n,\{\overline{q}\}} \left(\{\overline{z}_j\}\,; \overline{z}_0 \right), \tag{7}$$

with

$$H_n \varphi_{n,\{q\}} = \varepsilon_{n,\{q\}} \varphi_{n,\{q\}} \;\; , \;\; \overline{H}_n \overline{\varphi}_{n,\{\overline{q}\}} = \overline{\varepsilon}_{n,\{\overline{q}\}} \overline{\varphi}_{n,\{\overline{q}\}}\,. \tag{8}$$

The eigenvectors also satisfy the conformal invariance property [8]

$$\begin{aligned} &\chi_{n,\{q\}} \left(\{z_j, \overline{z}_j\}\,; z_0, \overline{z}_0 \right) \to \chi_{n,\{q\}} \left(\left\{ z_j', \overline{z}_j' \right\}; z_0', \overline{z}_0' \right) \\ &= (cz_0 + d)^{2h} \left(\overline{c}\overline{z}_0 + \overline{d} \right)^{2h} \chi_{n,\{q\}} \left(\{z_j, \overline{z}_j\}\,; z_0, \overline{z}_0 \right) \end{aligned} \tag{9}$$

under the transformations (4). The conformal weights h and $\overline{h}$ correspond to the principal series of $SL(2,\mathbf{C})$ [8]:

$$h = \frac{1+m}{2} - i\nu \quad , \quad \overline{h} = 1 - h^* = \frac{1-m}{2} - i\nu\,,$$

[1] For $n = 2$ and $n = 3$, as one looks for color neutral reggeons, the expression (2) is exact for any finite N_c. This is easily derived from the zero charge color condition $\sum_{j=1}^{n} t_j^a = 0$ [10].

with m integer and ν real.

It was shown in [5, 6] that H_n is the nearest-neighbour Hamiltonian of the spin zero XXX Heisenberg spin-chain with periodic boundary conditions. The spin s generators of $SL(2,\mathbf{C})$ at site j are:

$$S_j^+ = z_j^2 \partial_j - 2s z_j \ , \quad S^- = -\partial_j \quad , \quad S_j^3 = z_j \, \partial_j - s.$$

The R-matrix of the spin-s XXX chain acts in the tensor product space $h^{(s)} \otimes h^{(s)}$. It is given at spectral parameter λ, by [9]:

$$R_{12}^{(s)}(\lambda) = \frac{\Gamma(i\lambda - 2s)\Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J_{12})\Gamma(i\lambda + J_{12} + 1)} \ , \tag{10}$$

where the Casimir operator J_{12} satisfies the relation

$$J_{12}(J_{12} + 1) = (\vec{S}_1 + \vec{S}_2)^2 = 2\vec{S}_1.\vec{S}_2 + 2s(s + 1) \, . \tag{11}$$

One then has

$$H_{jk} = -i \frac{d}{d\lambda} \log R_{jk}^{(s=0)}(\lambda)_{|\lambda=0} \tag{12}$$

as can be seen from the representation (3). The matrix $R^{(s)}(\lambda)$ satisfies the Yang-Baxter equation. This implies that

$$H_n = \sum_{j=1}^{n} H_{j,j+1} \tag{13}$$

belongs to an infinite set of conserved quantities

$$\tau_k = -i \frac{d^k}{d\lambda^k} \log R^{(s=0)}(\lambda)_{|\lambda=0} \ , \ k = 0, 1, 2, ... \tag{14}$$

These quantities are simultaneously diagonalized by means of the algebraic Bethe Ansatz. In this approach one defines the Lax operators,

$$L_k^{(s)}(\lambda) = \lambda I_k \otimes I + i\vec{S}_k \otimes \vec{\sigma} = \begin{pmatrix} \lambda + iS_k^3 & iS_k^- \\ iS_k^+ & \lambda - iS_k^3 \end{pmatrix}$$

and the monodromy and transfer matrices:

$$T_a(\lambda) = L_n(\lambda)...L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \ , \ \Lambda(\lambda) = \mathrm{tr}\, T_a(\lambda) = A(\lambda) + D(\lambda) \tag{15}$$

The conserved quantities

$$\hat{q}_{n-k} = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Lambda(\lambda)_{|\lambda=0} \ , \quad k = 1, ..., n-2$$

commute with the operators τ_k because of the Yang-Baxter equation. One obtains

$$\Lambda(\lambda) = 2\lambda^n + \hat{q}_2\lambda^{n-2} + \hat{q}_3\lambda^{n-3} + \cdots + \hat{q}_n,$$

where

$$\hat{q}_k = \sum_{n\geq j_1>...>j_k\geq 1} i^k z_{j_1j_2}z_{j_2j_3}...\, z_{j_kj_1}\partial_{j_1}...\,\partial_{j_k}.$$

In particular we have the Casimir operator of the conformal algebra:

$$\hat{q}_2 = \sum_{n\geq j>k\geq 1} z_{jk}^2\partial_j\partial_k = -S^3S^3 - \frac{1}{2}\left(S^+S^- + S^-S^+\right) \equiv -h(h-1)\,.$$

A set of simultaneous eigenvectors of the $\hat{q}_k$ is also a set of simultaneous eigenvectors of the τ_k. Such vectors are then given by the algebraic Bethe Ansatz approach. A subclass can be written as:

$$\varphi^{(s=0)}_{n,\{\lambda_i\}}\left(\{z_j\}\,;0\right) = z_{12}z_{23}...z_{n1}B^{(s=-1)}(\lambda_1)...B^{(s=-1)}(\lambda_p)\frac{1}{z_1^2...z_n^2} \tag{16}$$

The appearance of the B-operators for spin -1 is due to the existence of a trivial highest-weight vector for the spin 0 chain. The spin -1 chain is related by a similarity transformation to the spin 0 chain. In particular the eigenvalues $q_k^{(s=0)}$ and $q_k^{(s=-1)}$ are equal. The parameters $\{\lambda_i\}$ satisfy the Bethe Ansatz equations for spin -1:

$$\left(\frac{\lambda_k - i}{\lambda_k + i}\right)^n = -\prod_{j=1}^{p}\frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}\,, \quad k = 1, ..., p\,. \tag{17}$$

The eigenvalues are given by

$$\varepsilon_n = -2n - 2\sum_{j=1}^{p}\frac{1}{\lambda_j^2 + 1}\,. \tag{18}$$

In fact it is possible to reformulate equations (17), (18) and (16) as:

$$\Lambda^{(s=-1)}(\lambda)Q_n(\lambda) = (\lambda + i)^nQ_n(\lambda + i) + (\lambda - i)^nQ_n(\lambda - i)\,, \tag{19}$$

$$\varepsilon_n = -2n + i\left(\frac{Q_n'(-i)}{Q_n(-i)} - \frac{Q_n'(i)}{Q_n(i)}\right), \tag{20}$$

$$\varphi^{(s=0)}_{n,\{\lambda_j\}}\left(\{z_j\};0\right) = z_{12}z_{23}...z_{n1}(iS^-)^{h-n}Q_n(x_1)...Q_n(x_{n-1})\frac{1}{z_1^2...z_n^2} \tag{21}$$

Equation (19) is the eigenvalue version of the Baxter equation; $\Lambda^{(s=-1)}(\lambda)$ and $Q_n(\lambda)$ stand for the operators and their eigenvalues. In this approach one looks for a solution $Q_n(\lambda)$ *analytical* in $\mathbf{C}$. In (21), $(x_1, ..., x_{n-1})$ are operators such that

$$B(\lambda) = iS^-(\lambda - x_1)...(\lambda - x_{n-1})\,, \quad S^- = -\sum_{j=1}^{n}\partial_j\,.$$

For a *polynomial* solution $Q_n(\lambda) = \prod_{j=1}^{p}(\lambda - \lambda_j)$ of the Baxter equation we obtain eqs. (16), (17) and (18). Thus the system (19), (20) and (21) gives a larger class of solutions than the class of polynomial solutions.

We conclude by rewriting the conformal invariance property using the generators $S^{\pm}, S^3$ of the whole chain, for spin zero:

$$\left(\sum_{j=1}^{n} \partial_j + \partial_0\right) \chi_{n\{q\}} = \left(\sum_{j=1}^{n} z_j\partial_j + z_0\partial_0 + h\right) \chi_{n,\{q\}}$$
$$= \left(\sum_{j=1}^{n} z_j^2\partial_j + z_0^2\partial_0 + h\right) \chi_{n,\{q\}} = 0\,,$$

and similarly with anti-holomorphic operators. This can be rewritten as

$$\begin{aligned} S^3\chi_{n,\{q\}}\left(\{z_j, \overline{z}_j\}\,;0,0\right) &= -h\,\chi_{n,\{q\}}\left(\{z_j, \overline{z}_j\}\,;0,0\right) \\ S^+\chi_{n,\{q\}}\left(\{z_j, \overline{z}_j\}\,;0,0\right) &= 0 \\ \chi_{n,\{q\}}\left(\{z_j, \overline{z}_j\}\,;z_0,\overline{z}_0\right) &= \chi_{n,\{q\}}\left(\{z_{j0}, \overline{z}_{j0}\}\,;0,0\right). \end{aligned}$$

Therefore $(-h)$ can be identified as the total spin of the vector $\chi_{n,\{q\}}$ and one must also have $-h = -1.n - p$, and thus $p = h - n$. This implies $h = n, n+1, ...$ for the polynomial solutions. There is also a $h \to 1-h$ symmetry in the problem which allows us to consider only $\mathrm{Re}\,h \geq \frac{1}{2}$.

3 The General Solution of the Baxter Equation

Following refs. [6, 10] we set

$$Q_n(\lambda) = \int_C \frac{dz}{2\pi i} z^{-i\lambda-1}(z-1)^{i\lambda-1}\tilde{Q}_n(z) \tag{22}$$

where the closed path C encircling the two points 0 and 1 counterclockwise is such that the integrand is uniform. The Baxter equation becomes an n^{th} order differential equation for $\tilde{Q}_n(z)$:

$$\left[\left(z(1-z)\frac{d}{dz}\right)^n + z(1-z)\sum_{k=0}^{n-2} i^{n-k} q_{n-k}\left(z(1-z)\frac{2}{dz}\right)^k\right]\tilde{Q}_n(z) = 0\,. \tag{23}$$

This is a Fuchsian differential equation with three regular singular points, $z = 0, 1$ and ∞ [11]. The indicial equations for these singular points are

$$s^n = 0 \;\text{ for }\; z = 0 \;\text{ and }\; z = 1\,,$$

$$s(s-1)...(s-n+3)\left[(s-n+2)(s-n+1) + q_2\right] = 0 \;\text{ for }\; z = \infty.$$

One can then look for logarithm-free solutions in the form of an entire series around any of the three singular points. We consider a series expansion around the point $z = 0$ of the form:

$$\tilde{Q}_n(z) = \sum_{k\geq 0} a_k z^k \tag{24}$$

It is then easy to see that the sequence $\{a_k\}$ will satisfy an n-term linear recurrence relation of the form

$$\sum_{i=0}^{n-1} a_{k+i}\, p_i(k+i) = 0\,, \tag{25}$$

where

$$p_{n-1}(k) = k^n\,,\ p_0(k) = k(k+1)...(k+n-3)\left((k+n-2)(k+n-1)+q_2\right)\,,$$

and for all i, $p_i(k)$ is a polynomial of degree n in k with a linear dependence on the parameters $\{q_k\}$.

Expansions around $z=1$ and $z=\infty$ lead to similar recurrence relations. An expansion on the basis of the Legendre polynomials was considered in ref. [10, 12].

Deforming the integration contour in eq. (22), one can rewrite $Q_n(\lambda)$ as

$$Q_n(\lambda) = i\frac{\sinh(\pi\lambda)}{\pi}\int_0^1 x^{-i\lambda-1}(1-x)^{i\lambda-1}\tilde{Q}_n(x)dx \tag{26}$$

Plugging the series expansion (24) into eq. (26) one obtains:

$$Q_n(\lambda) = \sum_{k=1}^{\infty}\frac{a_k}{(k-1)!}\prod_{l=1}^{k-1}(l-i\lambda) \tag{27}$$

Thus a polynomial solution of degree p in z of the differential equation (23) yields a polynomial solution, of degree $p-1$ in λ, to the Baxter equation. It is possible to verify directly that, at least for the first few values of n, the expression (27) provides a solution of the Baxter equation if the recurrence (25) is satisfied. To do so one just expands the polynomials λ^j and $(\lambda\pm i)^n$ in a way which is compatible with an expansion over the basis provided by

$$P_0(\lambda)=1\,,\ \ P_k(\lambda)=\prod_{l=1}^{k-1}(l-i\lambda)\ \ ,\ \ k=1,2,...$$

We note here that for any periodic function f of λ with period i, one can define a new solution of the Baxter equation by multiplying any solution with $f(\lambda)$.

We now briefly address the issue of the convergence of a series of the type (27). For $n=2$ the solution recessive at 0 of the differential equation (23) is the hypergeometric function

$$\tilde{Q}_2(z) = {}_2F_1(h,1-h;1;z)$$

for $h\notin \mathbf{Z}$. For k large, $a_k\sim\frac{1}{k}+\mathcal{O}\left(\frac{1}{k^2}\right)$ and the series (27) converges absolutely and uniformly for $\mathrm{Im}\,\lambda<0$, where $Q_2(\lambda)$ is therefore analytical. The Baxter equation allows to extend analytically $Q_2(\lambda)$ to $\mathbf{C}\backslash i\mathbf{N}$.

There is generically an infinite number of poles at the points $0, i, 2i, ...$ By calculating

$$Q_2(\lambda)(2\lambda^2+q_2)-Q_2(\lambda-i)(\lambda-i)^2\,,$$

and its derivative at $\lambda = 0$, with the help of the recurrence relation, we find that the pole at 0 is simple with residue $\frac{-i}{(h-1)!(-h)!}$. The Baxter equation implies then the existence of other simple poles at $i, 2i, ...$

We also found numerically that $Q_2(\lambda)$ increases exponentially as $\lambda \to -i\infty$ for $h = 1/2$ and $h = 5/2$; we believe that this is the case for generic value of h. If one assumes $Q_2(\lambda) \sim \lambda^\alpha$ as λ tends to *real* infinity then the Baxter equation implies that $\alpha = h - 2$.[2] Numerical trials confirm this assumption.

The other solution of the differential equation, ${}_2F_1(h, 1-h; 1; 1-z)$, gives the series $Q_2(-\lambda)$ which is therefore also a solution of the Baxter equation with the same value of q_2. One way to construct a solution of the Baxter equation analytical in $\mathbf{C}$ is to consider

$$\sinh(2\pi\lambda)\left(c_1 Q_2(\lambda) + c_2 Q_2(-\lambda)\right)$$

for any values of c_1 and c_2. The behaviour of this analytical solution at real infinity is not likely to be λ^{h-2} however. In this respect we seem to disagree with the results of [6, 10].

We now consider $n = 3$, the first "non-trivial" case. The recurrence (25) becomes:

$$\begin{aligned} a_{k+2}(k+2)^3 - a_{k+1}\left[iq_3 + (k+1)\left(q_2 + (2k+3)(k+2)\right)\right] \\ +a_k k\left((k+1)(k+2) + q_2\right) = 0 \;\text{ for }\; k = -1, 0, 1, ..., \;\text{ with }\; a_{-1} \equiv 0\,. \end{aligned} \tag{28}$$

It does not seem possible to find an explicit solution to this recurrence. However, because the indicial equations at $z_0 = 0$ and $z_0 = 1$ are $s^3 = 0$, the general solution of the differential equation around these points is of the type $f_1(z) + f_2(z)\log(z - z_0) + f_3(z)\log^2(z - z_0)$ where $f_i(z)$ are regular functions at $z = z_0$. The only possible singularities of the solutions are the three singular points. Thus $z = 1$ is the only singularity on the circle of radius one of the solution (24). The logarithmic behaviour of the solution around one implies then, using the Darboux theorem, the following behaviour of a_k as k tends to infinity:

$$a_k = \left(\frac{\alpha_1}{k} + \frac{\alpha_2}{k^2} + \cdots\right)\log k + \frac{\beta_1}{k} + \frac{\beta_2}{k^2} + \cdots \tag{29}$$

From the recurrence equation, it is possible to find relations between the coefficients appearing in (29). Numerical trials confirm this logarithmic behaviour for ka_k as k becomes large. Such a behaviour allowed us to show that for $\mathrm{Im}\,\lambda < 0$ the series (27) converges uniformly. One then extends analytically this solution to almost all the complex plane through the Baxter equation. Again, proceeding similarly to the $n = 2$ calculation, the points $i\mathbf{N}$ turn out to be poles. They are of order two. For instance we find for λ near 0

$$Q_3(\lambda) \sim -\frac{\alpha_1}{\lambda^2}\,.$$

For the solution $Q_{3\{q_2,q_3,\}}(\lambda)$ just defined one then considers the function $Q_{3\{q_2,-q_3\}}(-\lambda)$. This is also a solution of the Baxter equation with (q_2, q_3) as parameters and

$$\sinh^2(2\pi\lambda)\left(c_1 Q_3\{q_2, q_3\}(\lambda) + c_2 Q_3\{q_2, -q_3\}(-\lambda)\right) \tag{30}$$

[2] Note however that an analytical function in $\mathbf{C}$ cannot have a branch-type singularity of the form $\lambda^{h-n} (h \notin \mathbf{Z})$ as λ tends to complex infinity.

is also a solution, which is analytical in $\mathbf{C}$. The behaviour at infinity is not likely to be simple.

For any given value of n, one can in principle repeat the same analysis and expect to find poles of order $n-1$ at $i\mathbf{N}$ for the analytically extended series solution.The approach of [13] to constructing solutions of the Baxter equation of the Toda chain with a given behaviour at infinity, can also be tried. We are investigating this possibility.

4 Polynomial solutions of the Baxter Equation

Polynomial solutions to the Baxter equation provide a subclass of eigenvectors of the operators $\hat{q}_2, \hat{q}_3$ and H_n.

It is possible to obtain polynomial solutions from the series (27) by requiring the sequence $\{a_k\}$ to truncate. For a polynomial of degree p it is necessary to have $a_{p+i} = 0$ for all $i \geq 2$. However, for a given n, the sequence satisfies the n-term recurrence relation (25) and it is enough to require

$$a_{p+1} \neq 0 \ \text{ and } \ a_{p+2} = a_{p+3} = \cdots = a_{p+n} = 0\,.$$

These relations are enough to quantize the eigenvalues $q_2, ..., q_n$. For instance setting $k = p+1$ in (25) one obtains

$$q_2 = -(p+n)(p+n-1) = -2\begin{pmatrix} p+n \\ 2 \end{pmatrix} \tag{31}$$

Writing $q_2 = -h(h-1)$ gives $h = p+n$ as was found in [6, 10]. The conditions on $q_3, ..., q_n$ are given by the roots of a coupled system of polynomial equations in the variables q_i's. For $n = 3$ the discretized values of q_3 are given by the roots of a degree $p+1$ polynomial.[3] It turns out that all the roots seem to be real, in accordance with the results of [6, 10].

We now derive equations for the invariants of polynomial solutions to the Baxter equation. Let

$$\sigma_q(X_i) \equiv \sigma_q(X_1, ..., X_p) \equiv \sum_{1\leq r_1<r_2<\cdots<r_q\leq p} X_{r_1}...X_{r_q}\,, \quad q = 1, ..., p, \tag{32}$$

$$\sigma_0(X_i) \equiv 1 \tag{33}$$

be the elementary symmetric polynomials. The polynomials $\sigma_q(X_i)$ satisfy the relations

$$\sigma_q(X_i + a) = \sum_{r=0}^{q} \begin{pmatrix} p-q+r \\ p-q \end{pmatrix} a^r \sigma_{q-r}(X_i) \tag{34}$$

A polynomial solution $Q_n(\lambda)$ of the Baxter equation, whose roots therefore satisfy the Bethe Ansatz equations, can be written:

$$Q_n(\lambda) = \prod_{j=1}^{p}(\lambda - \lambda_j) = \sum_{j=0}^{p}(-1)^{p-j}\sigma_{p-j}\lambda^j$$

[3] *Mathematica* and *Maple* give exact roots up to $h = 15$, and fail otherwise.

where

$$\sigma_j \equiv \sigma_j(\lambda_1, ..., \lambda_p)\,.$$

We then rewrite the Baxter equation as a set of equations:

$$\sum_{k=0}^{n-2}(-1)^{n-1+k} q_{n-k}\sigma_{p-m+k} = 2\sum_{k=0}^{\left[\frac{p-m+n-2}{2}\right]}(-1)^k \begin{pmatrix} m+2k+2 \\ m \end{pmatrix} \sigma_{p-m+n-2-2k}\,, \qquad (35)$$

$$\begin{aligned} 0 &\leq m \leq p+n-2\,, \qquad \sigma_{-n+2} = \cdots = \sigma_{-1} = 0\,, \\ \sigma_0 &= 1\,, \quad \sigma_{p+1} = \cdots = \sigma_{p+n-2} = 0. \end{aligned}$$

where $[x]$ is the integer part of x. The p equations, $m = 1, ..., p$, give a triangular linear system, with parameters $(q_3, ..., q_n)$, for the unknowns $(\sigma_1, ..., \sigma_p)$. The equation $m = p+n-2$ gives, as expected, equation (31). The $n-2$ remaining equations, $m = 0$ and $m = p+1, ..., p+n-3$, give the quantization conditions, once the σ_k's are solved for as polynomials in the parameters $(q_3, ..., q_n)$.

For $n = 2$, we get that the polynomial $Q_2(\lambda)$ is even (odd) for p even (odd). For $n = 3$, the quantization condition for q_3 can be written as a vanishing determinant of a matrix with a lower triangular part and one non-vanishing line above the diagonal, once σ_0 is introduced in the set of unknowns.[4]

The energy ε_n of the Hamiltonian (13) can be rewritten as follows:

$$\begin{aligned} \varepsilon_n &= -2n + i\left(\frac{Q'_n(-i)}{Q_n(-i)} - \frac{Q'_n(i)}{Q_n(i)}\right) && (36) \\ &= -2n - 2\sum_{k=1}^{p}\frac{1}{\lambda_k^2+1} && (37) \\ &= -2n + i\left(\frac{\sigma_{p-1}(\lambda_k - i)}{\sigma_p(\lambda_k - i)} - \frac{\sigma_{p-1}(\lambda_k + i)}{\sigma_p(\lambda_k + i)}\right). && (38) \end{aligned}$$

One can then use relation (34) to rewrite ε_n in terms of the σ_k's. Equations (35) provide a *new* way for looking at the Bethe Ansatz equations. It is also possible to obtain similar equations for any spin s, not just the case $s = -1$ under consideration.

5 A new expression for the eigenvectors

In the framework of the algebraic Bethe Ansatz, the eigenvectors are obtained as the repeated action of a "lowering" operator $B(\lambda)$ on a highest weight state, or pseudo-vaccum state, as in equations (16) and (21). However both expressions are unwieldy as the number of Bethe Ansatz roots increases, or for non-integer values of h. We now develop compact expressions for the eigenvectors as linear combinations of eigenvectors of the operator $\hat{q}_2$.

[4] We checked using *Mathematica* that, for p up to 15, the values of q_3 coincide with those obtained from the recurrence relation for $n = 3$.

Define the vectors

$$\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_1,z_2,z_3) \equiv z_1^{\alpha_1-h} z_2^{\alpha_2-h} z_3^{\alpha_3-h} z_{12}^{\alpha_3} z_{23}^{\alpha_1} z_{31}^{\alpha_2} \,, \tag{39}$$

where

$$\alpha_1\,,\ \alpha_2\,,\ \alpha_3\ \in \mathbf{C}\,,\ \ \alpha_1+\alpha_2+\alpha_3 = h\,. \tag{40}$$

In what follows we always assume (40) to hold. The form of the vector $\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_1,z_2,z_3)$ is suggestive of a generalization of the pomeron ($n=2$) eigenvectors:

$$\varphi_h(z_1,z_2;z_0) = \left(\frac{z_{12}}{z_{10}z_{20}}\right)^h \,. \tag{41}$$

One can directly verify that the functions (39) satisfy the conformal invariance property (9), or by viewing it as a four-point function of a conformally invariant theory [14]. Such four-point functions were considered by Lipatov in a different approach [8]. These functions are eigenvectors of

$$\hat{q}_2 = \sum_{j=1}^{3} z_{ii+1}^2 \partial_i \partial_{i+1} \,,$$

with eigenvalues $-h(h-1)$. One also has the linear dependence relations

$$\varphi_{\alpha_1-1,\alpha_2+1,\alpha_3}(z_i) + \varphi_{\alpha_1-1,\alpha_2,\alpha_3+1}(z_i) = -\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_i) \tag{42}$$

with two other equivalent relations where $\alpha_1-1 \to \alpha_2-1$ or $\alpha_1-1\to\alpha_3-1$. Other more complicated relations exist. Note that (42) is satisfied without the constraint (40).

The action of $\hat{q}_3$ on $\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_i)$ is a tedious but straightforward calculation. We obtain:

$$\begin{aligned} i\hat{q}_3\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_i) &= (\alpha_3\alpha_2(\alpha_2-1)+\alpha_2\alpha_1(\alpha_1-1) \\ &+ \alpha_1\alpha_3(\alpha_3-1))\,\varphi_{\alpha_1,\alpha_2,\alpha_3}(z_i) \\ &- \alpha_1(\alpha_1-1)(\alpha_1-h)\varphi_{\alpha_1-1,\alpha_2+1,\alpha_3}(z_i) \\ &- \alpha_2(\alpha_2-1)(\alpha_2-h)\varphi_{\alpha_1,\alpha_2-1,\alpha_3+1}(z_i) \\ &- \alpha_3(\alpha_3-1)(\alpha_3-h)\varphi_{\alpha_1+1,\alpha_2,\alpha_3-1}(z_i) \end{aligned} \tag{43}$$

The action of $\hat{q}_3$ is that of a step operator. It is therefore natural to look for simultaneous eigenvectors of $\hat{q}_2$ and $\hat{q}_3$ (and therefore of H_3) as linear combinations of the eigenvectors of $\hat{q}_2$, that is, as a kind of coherent states.

The eigenvalues of $\hat{q}_2$ are highly degenerate: for a fixed value of h, two complex parameters label this degeneracy. In order to control such degeneracy, we first consider the plane determined by (40) in $\mathbf{C}^3$, and fix a point $(\alpha_1,\alpha_2,\alpha_3)$ in it. The set

$$\varphi_{\alpha_1,\alpha_2+m,\alpha_3-m}\,,\ \ \varphi_{\alpha_1-n,\alpha_2,\alpha_3+n}\,,\ \ \varphi_{\alpha_1+p,\alpha_2-p,\alpha_3}\,,\ \ m,n,p\in\mathbf{N}\,,$$

is a basis for the space

$$\varphi_{\alpha_1+m,\alpha_2+n,\alpha_3+p}\,, \quad m,n,p \in \mathbf{Z}\,, \quad m+n+p=0\,.$$

This space can be represented as shown in figure 1, where the vertices correspond to the foregoing vectors. The vertices on the three solid lines are the basis vectors. The triangle in bold lines represents a three-term linear dependence relation of the kind (42). This diagram is reminiscent of the $sl(3)$ weight lattice.

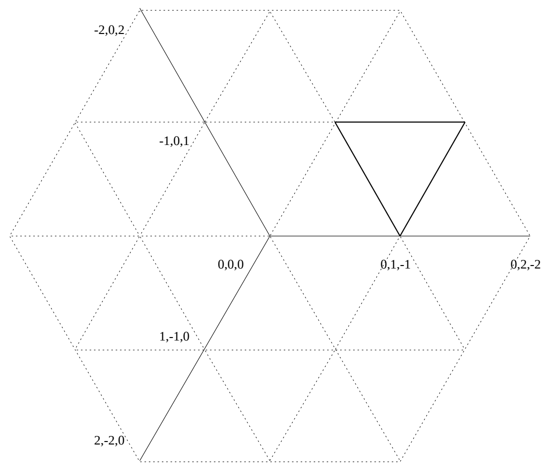

Figure 1: Visualization of the space with origin $(\alpha_1,\alpha_2,\alpha_3)$, represented as $(0,0,0)$.

We can then look for eigenvectors of $\hat{q}_3$ as a linear combination of basis vectors,

$$\phi_{q_3}(z_i) = \sum_{m\geq 0} a_m \varphi_{\alpha_1,\alpha_2+m,\alpha_3-m} + \sum_{n\geq 0} b_n \varphi_{\alpha_1-n,\alpha_2,\alpha_3+n} + \sum_{p\geq 0} c_p \varphi_{\alpha_1+p,\alpha_2-p,\alpha_3}\,.$$

Requiring $\phi_{q_3}(z_i)$ to be an eigenvector of $\hat{q}_3$ gives a complicated set of coupled recurrence relations with an infinite number of terms. To obtain a more manageable set of relations we restricted the point $(\alpha_1,\alpha_2,\alpha_3)$ to one of the three points $(h,0,0)$, $(0,h,0)$ and $(0,0,h)$. The recurrence relations then decouple and become three-term recurrence relations of the type already encountered in section 3. We now give these relations for the point $(h,0,0)$. We note that for integer h it is enough to consider this point. Consider the vector

$$\phi_{h,q_3}(z_i) = \sum_{m\geq 1} a_m \varphi_{h,m,-m}(z_i) + \sum_{m\geq 1} b_m \varphi_{h-m,0,m}(z_i) + \sum_{m\geq 1} c_m \varphi_{h+m,-m,0}(z_i)\,. \quad (44)$$

We have not considered the three $m=0$ terms in this sum because such terms are in the kernel of $\hat{q}_3$ and we are interested in non-vanishing eigenvalues. The action of $\hat{q}_3$ on (44) is easy to obtain from (43). We get three uncoupled recurrence relations for the coefficients in the sum which means that one can consider each sum by itself as an eigenvector. The recurrence equations are:

$$(m+1)m(m+1-h)a_{m+1} \quad + \quad \Big(iq_3 + m(2m^2-h(h-1))\Big)\, a_m$$

$$
\begin{aligned}
&+ \quad m(m-1)(m+h-1)a_{m-1} = 0\,, && (45)\\
(m+1)m(m+1-h)b_{m+1} &+ \quad (iq_3 + m(m-h)(2m-h))b_m \\
&+ \quad (m-1)(m-h)(m-1-h)b_{m-1} = 0\,, && (46)\\
(m+1)(m+1+h)(m+h)c_{m+1} &+ \quad (iq_3 + m(m+h)(2m+h))c_m \\
&+ \quad m(m-1)(m-1+h)c_{m-1} = 0\,. && (47)
\end{aligned}
$$

These recurrence relations are different from eq. (28) but of the same type. Polynomial solutions of the Baxter equation correspond to finite sums in (44). For instance, the solutions

$$Q_3(\lambda) = \lambda \pm \frac{1}{\sqrt{3}}$$

with $q_3 = \pm 2\sqrt{3}$ and $q_2 = -12$ ($h = 4$), yield the two eigenvectors:

$$
\begin{aligned}
\phi_{4,\pm 2\sqrt{3}}(z_i) &= \mp\sqrt{3}\varphi_{121} + i(\varphi_{112} - \varphi_{211}) && (48)\\
&= \mp\sqrt{3}(\varphi_{103} + \varphi_{301} + 2\varphi_{202}) + i(\varphi_{301} - \varphi_{103})\,. && (49)
\end{aligned}
$$

The first expression is calculated directly from (16), where

$$
\begin{aligned}
B^{(-1)}(\lambda) &= i\lambda^2(S_1^- + S_2^- + S_3^-) + \lambda(S_2^- S_1^3 - S_2^3 S_1^- + S_3^- S_1^3 - S_3^3 S_1^- \\
&+ \quad S_3^- S_2^3 - S_3^3 S_2^-) - i(S_3^- S_2^+ S_1^- + S_3^- S_2^3 S_1^3 + S_3^3 S_2^3 S_1^- - S_3^3 S_2^- S_1^3)
\end{aligned}
$$

with spin -1 generators.

Equation (49) is obtained from equation (48) by using the relations (42). It is also obtained from (44) and the recurrence relation (46). More generally, the relations (45–47) can be truncated to a finite number of terms by requiring $h = -n_0$ for (45) and (47), $h = n_0 + 1$ for (46) ($q_2 = -h(h-1) = -n_0(n_0+1)$ for the three cases), and the relations for $n = n_0$ to hold. We verified that the discretized values of q_2 and q_3 we obtain are the same as the values we found in sections 3 and 4, for many values of h. We also found an additional $q_3 = 0$ solution for the recurrences (45) and (46).

Eigenvectors common to $\hat{q}_2$ and $\hat{q}_3$ and with vanishing eigenvalue q_3 are just linear combinations of pomeron eigenvectors. This becomes clear if one solves the partial differential equation

$$\hat{q}_3\varphi = iz_{12}z_2z_{31}\partial_1\partial_2\partial_3\varphi = 0\,;$$

it implies that $\varphi(z_i)$ is a sum of three functions which depends on only two of the three variables (z_1, z_2, z_3). For instance one has :

$$\varphi_{h,0,0}(z_{10}, z_{20}, z_{30}) = \varphi_h(z_2, z_3; z_0)\,,\ \varphi_{111}(z_{i0}) = \frac{1}{3}(\varphi_{003} + \varphi_{030} + \varphi_{300})(z_{i0})\,.$$

The action of H_3 on $\varphi_{q_3=0}$ is straightforward because of the foregoing remarks:

$$H_3\varphi_{q_3=0} = H_2(q_2)\varphi_{q_3=0}\,.$$

However the action of H_3 on $\varphi_{q_3\neq 0}$ is not simple. For polynomial solutions, corresponding to truncated sums in (44), we obtain the eigenvalue of H_3 from (36), (37) or (38).

The Ansatz (44) covers all the polynomial solution class of the Baxter equation. We also believe it provides a space rich enough to cover most of the non-polynomial solutions we are interested in.

An extension of the foregoing approach to a chain with four sites and larger sizes can be considered. For four sites, the "elementary" functions, which are eigenvectors of $\hat{q}_2$ with eigenvalue $q_2 = -h(h-1)$, are:

$$\varphi_{q_2,\{\alpha\}}(z_i) = z_1^{\alpha_1-h} z_2^{\alpha_2-h} z_3^{\alpha_3-h} z_4^{\alpha_4-h} z_{12}^{\alpha_{34}-h} z_{23}^{\alpha_{14}-h} z_{34}^{\alpha_{12}-h} z_{41}^{\alpha_{23}-h} z_{13}^{\alpha_{24}-h} z_{24}^{\alpha_{13}-h}\,, \tag{50}$$

with

$$\begin{aligned} h &= \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} + \alpha_{34}\,, \\ \alpha_1 = h - \alpha_{23} - \alpha_{24} - \alpha_{34}\,, &\quad \alpha_2 = h - \alpha_{13} - \alpha_{14} - \alpha_{34}\,, \\ \alpha_3 = h - \alpha_{12} - \alpha_{14} - \alpha_{24}\,, &\quad \alpha_4 = h - \alpha_{12} - \alpha_{13} - \alpha_{23}\,. \end{aligned}$$

The actions of $\hat{q}_3$ and $\hat{q}_4$ can be calculated, and these operators are then seen to act as step operators. One can then take an Ansatz for the eigenvectors and find the recurrence relations for the coefficients. The generalization of the elementary functions (50) to larger chains is straightforward.

6 Conclusion

The determination of the ground state of the Hamiltonian H_n for $n \geq 3$ is still an open problem. In this paper we analyzed the Baxter equation connected with the diagonalization. Common features emerged such as the series solutions and the n-term recurrence associated with them. It seems impossible to reduce such recurrences to simpler ones. Such recurrence relations emerged again when we looked for a simple form for the eigenvectors. We also recast the Baxter equation as a linear system suitable for the search for polynomial solutions.

Acknowledgements: It is a pleasure to thank V. Pasquier, M. Bauer, R. Peschanski and S. Nonnenmacher for many useful discussions. We would like to thank G. Korchemsky for fruitful discussions. S. W. is grateful to L. N. Lipatov and J. Bartels for discussions. Special thanks to J. B. Zuber for bringing to our attention the deep $\frac{SW}{ZM}$ mirror symmetry present in this problem, and for a careful reading of the manuscript.

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